Use of Lambert W function in determining speed for macroscopic traffic flow models

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Abstract

Computation of the speed for a specified traffic flow is often complicated particularly for cases where the speed-flow relationship is expressed in a logarithmic or an exponential form. To address the issue, this paper presents analytical formulae for computing speed in context of some established traffic flow models. The formulae proposed have been derived by adopting the concept of Lambert W function. Two branches of Lambert W have been judiciously utilized for this purpose. The approach suggested in this study, is capable of finding the speed both for the uncongested and the congested condition at a given traffic flow. Moreover, the proposed approach is simple to use and hence is suitable for on-field application.

Keywords:Speed; Flow; Lambert W; Traffic flow model.

1. Introduction

Speed, flow and density and their inter-relations are extremely important for defining a traffic stream and estimating capacity of a road section. Hence, researchers across the globe have devoted significant attention towards developing traffic flow models. The pioneering work in this aspect was carried out by Greenshields et al. (1935) who recommended a linear speed-density relationship. Later, Greenberg (1959) and Underwood (1961) respectively proposed a logarithmic and an exponential model that depict the speed-density relation. However, the common approach shared by all the above mentioned models was to calibrate the model by using speed and density data and then to convert it into a speed-flow model (Lum et al. 1998). These speed flow models are generally used to estimate speed based on the traffic flow.

One issue associated with the speed-flow model is its computational complexity. Apart from the Greenshields model, the other two models (Greenberg and Underwood models) are difficult to solve by using conventional analytical approach. Naturally, one has to rely on either an iterative or a graphical approach. While iterative approaches are

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highly sensitive to the initially assumed solution, speeds predicted using the graphical approach are often inaccurate. Moreover, both these approaches are tedious after all. Hence, it is essential to develop an efficient method for estimating speed from the speed-flow models.

The present research provides an efficient method for computing speed from the speed-flow models. To this end, the concept of Lambert W function has been introduced in the context of traffic flow modelling. The visible outcomes of this work are two analytical formulae for computing speed based on Greenberg and Underwood models.

2. Problem statement

The prime use of a speed-flow-density model can be observed in capacity estimation of a road section. Additionally, it is also utilized for predicting speed corresponding to a given traffic flow or density. In this regard, it is widely acknowledged that the speed-flow model is handier and more widely adopted over the speed-density model. This is because field measurement of flow is significantly simpler and cost effective as compared to that of density (Dhamaniya and Chandra 2013; Thomas et al. 2012). Few fundamental traffic flow models and their usefulness in speed determination can be instanced in this regard. For example, Equation-1 is the Greenberg's speed-flow model.

$$q = k_j v e^{-\left(\frac{v}{v_0}\right)} \tag{1}$$

where q and v are the two variables representing the traffic flow and the speed respectively. The two constants namely, jam density k_j and speed at capacity v_0 are generally known. Similarly, the underwood model is represented as

$$q = -k_0 v \ln\left(\frac{v}{v_f}\right) \tag{2}$$

wherev_f and k_0 are the constants respectively denoting the free flow speed and the density at capacity. It is to be noted that it is difficult to obtain analytical solutions of Equation-1 and 2 by using conventional approaches.

One possible option for solving Equations-1 and 2 is by utilizing iterative approach. The iterative approach entails (a) selecting an initial solution and (b) updating the solution until convergence. However, this approach is highly sensitive to the assumed initial solution. Additionally, the iterative approach is tedious and often requires significant computational time.

The other possible alternative is to employ graphical method. In this method, speed corresponding to a flow is estimated by using a graph that depicts the possible variation of speed with flow. However, the graphical method is comparatively tedious and the accuracy of the result might get compromised. Therefore, there is a need to find an alternative approach for computing speed based on the speed flow model.

3. Lambert W function

Suppose, two variables x and z be related as:

$$xe^x = z \tag{3}$$

Euler (1927) introduced a function for solving the Equation-3 and represented it as:

$$x = W(z) \tag{4}$$

Since the work by Euler was an extension of the pioneer research of Lambert who previously obtained a solution of the trinomial equation $x=q+x^m$(Lambert 1758), the function W was named as Lambert W in his honour. W(z) can be computed as:

$$W(z)=\sum_{n=1}^{\infty}\frac{(-n)^{n-1}}{n!}z^n \tag{5}$$

Lambert W function has two branches: the principal branch (W(z) > -1) and the minor branch (W(z) < -1). The function changes its branch at the point (-1/e, -1) as shown in Figure 1. Further, it is quite interesting to see how W(z) changes within $-1/e < z < 0$. Within this specified range of z, W(z) is double valued while it is single valued for z > 0 and $z = -e^{-1}$ (Barry and Culligan-Hensley 1995).

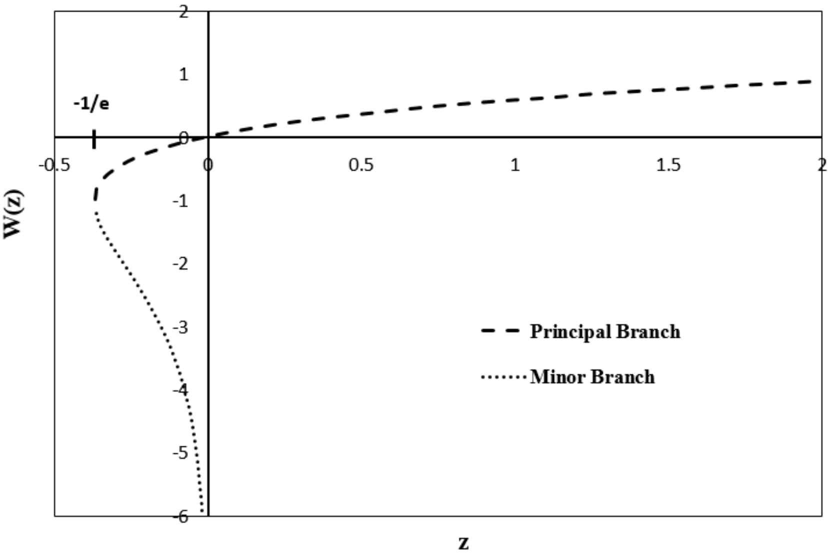

Figure 1: Schematic diagram representing two branches of Lambert W function.

It is worthwhile to mention that although wide applications of Lambert W function can be found in characteristics of solar cell (Jain and Kapoor 2004), black body radiation (Valluri et al. 2000), chemical kinetics (Williams 2010) etc., this concept have never been utilized in the context of traffic flow model. This is the first instance where the Lambert W function has been used in the context of traffic flow modelling.

4. Application of Lambert W in traffic flow models

The following discussion explains the application of Lambert W function in solving Greenberg and Underwood models.

4.1 Application on Greenberg model

In order to represent v as a function of q, Equation-1 is rewritten as

$$-\frac{q}{k_j v_0}=-\frac{v}{v_0}e^{\left(-\frac{v}{v_0}\right)} \tag{6}$$

Now, assuming

$$A = -\frac{q}{k_j v_0} \tag{7}$$

and

$$B = -\frac{v}{v_0} \tag{8}$$

Equation-6 can be expressed as

$$A = B e^{B} \tag{9}$$

Equation-9 has similar functional form as Equation-3. Hence, the solution of Equation-9 is

$$B = W(A) \tag{10}$$

where $W(\bullet)$ is the Lambert W function. Substituting A and B in Equation-10

$$-\frac{v}{v_0} = W\left(-\frac{q}{k_j v_0}\right) \tag{11}$$

Therefore speed (v) can be expressed as

$$v = -v_0 W\left(-\frac{q}{k_j v_0}\right) \tag{12}$$

Equation-12 is the solution of Greenberg’s speed-flow model and can be used to determine the speed for a given traffic flow.

4.2 Application on Underwood model

Similar kind of modification can be implemented upon Underwood model also. In order to represent v as a function of q, Equation-2 is rewritten as

$$-\frac{q}{k_0 v_f} = \frac{v}{v_f} \ln\left(\frac{v}{v_f}\right) \tag{13}$$

Now, assuming

$$C = -\frac{q}{k_0 v_f} \tag{14}$$

and

$$D = -\frac{v}{v_f} \tag{15}$$

Equation-13 can be expressed as

$$C = D \ln(D) \tag{16}$$

Considering another variable M such that

$$M = \ln(D) \tag{17}$$

Hence, Equation-16 now becomes

$$C = M e^{M} \tag{18}$$

Equation-16 has similar functional form as Equation-3. Hence, the solution of Equation-16 is

$$M = W(C) \tag{19}$$

From Equation-16,

$$D = \frac{C}{\ln(D)} \tag{20}$$

Combining Equations-17, 19 and 20

$$D = \frac{C}{W(C)} \tag{21}$$

Finally, substituting C and D in Equation-21 and simplifying

$$v = -\frac{q}{k_0 \, W\left(-\frac{q}{k_0 v_f}\right)} \tag{22}$$

Equation-22 is the solution of Underwood's speed-flow model and can be used to determine the speed at a given traffic flow.

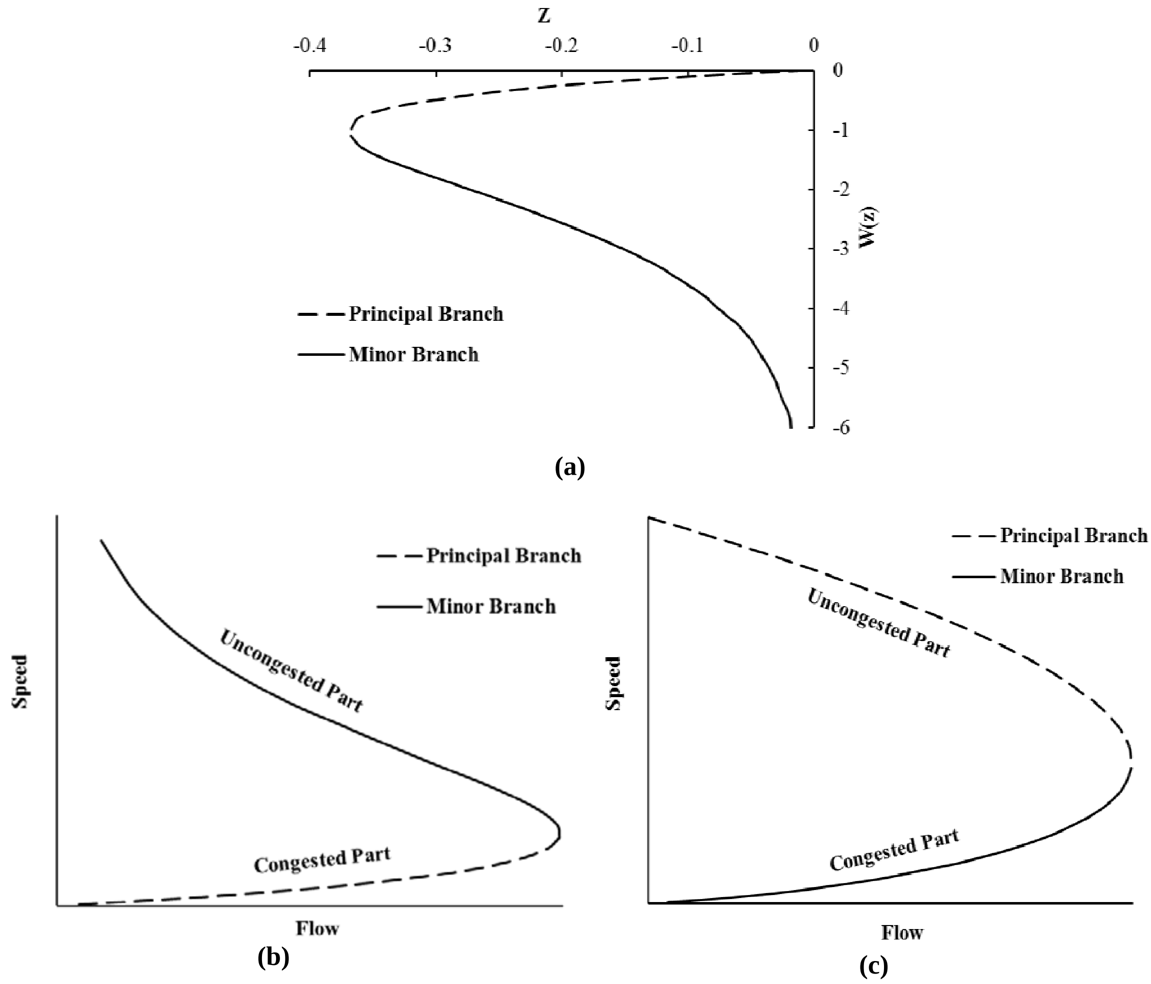

Figure 2: (a) Branches of Lambert W function and their uses in (b) Greenberg and (c) Underwood model

It is to be noted that the component inside the parentheses in both the formulae proposed (Equations-12 and 22) is negative. Therefore, Lambert W function in each of these cases, has two branches (principal and minor) and accordingly yields two negative solutions as shown in Figure 1. This is helpful in determining the speed corresponding to both the traffic flow conditions: uncongested and congested. In the case of Equation-12 (derived from Greenberg model), the minor branch of the lambert W function gives the speed corresponding to the uncongested part of the speed-flow curve whereas the principal branch yields the speed in the congested part. In Equation-22 (derived from Underwood model), on the other hand, the principal branch of the lambert W function corresponds to the uncongested part while the minor branch is related the congested part of the curve. Both of these cases are illustrated in Figure 2. Furthermore, in order to reduce the computational efforts of the users and to make this approach more accessible, Lambert W values corresponding to the possible range of input variable both for the principal and the minor branch, have been provided in Appendix.

5. Example Problem

For illustrating the above concept we assume a relationship between speed (v, km/h) and flow (q, veh/h) of traffic on a highway section as given below.

$$q = -125 \text{ v } \ln\left(\frac{v}{60}\right) \tag{23}$$

It is now required to compute the speed corresponding to a flow of 2000 veh/h.
The step wise solution to this problem is given below:

Step 1: Equation-23 is an Underwood speed-flow model. With reference to the general model (Equation-2), following parameters are determined.
Free flow speed (v_f) = 60 km/h
Density at capacity (k_0) = 125 veh/km

Step 2: The values of these parameters are put in Equation-22 which is the speed prediction equation derived from Underwood model. Therefore, the speed at a given flow (q) of 2000 veh/h

$$v = -\frac{2000}{125 \times W\left(-\frac{2000}{125 \times 60}\right)}$$

$$v = -\frac{2000}{125 \times W(-0.267)} \tag{24}$$

Step 3: To determine W(-0.267), Tables 1 and 2 in Appendix are to be used.

$$z = -0.267; \quad x = -0.26 \text{ and } \quad y = 7$$

In Tables 1 and 2, the cell where the row x = -0.26 and the column y = 7 are intersecting, is identified and the corresponding cell value is the required W(-0.267).
Hence for the principal branch, $W(-0.267) = -0.3972$
and for the minor branch, $W(-0.267) = -2.0271$

Step 4: The value of W(-0.267) is placed in Equation-24. For Underwood model, the principal and the minor branch characterise the uncongested and the congested part of the speed-flow curve respectively (Figure 2). Therefore, the speed at a flow of 2000 veh/h is obtained as below.

For uncongested condition $v = 40.28 \text{ km/h}$

For congested condition $v = 7.89 \text{ km/h}$

The results obtained by this approach can be verified by substituting the values of v in the given model (Equation-23).

6. Conclusions

An analytical approach to determine the speed for two well established speed-flow models is presented in this paper. To this end, the concept of Lambert W function has been introduced in the context of traffic flow modelling. Two analytical formulae corresponding to Greenberg model and Underwood model, have been proposed for computing speed at a given traffic flow. One additional advantage of the proposed formulae resides in its capability to compute speed corresponding to both congested and uncongested traffic conditions. However, the scope of the present work is limited to logarithmic and exponential traffic flow models only. Further research is required to address the similar problems appearing in other macroscopic traffic flow models.

Appendix: Lambert W values

Table 1: Values of W(z) for the principal branch

$$z = x - \left(y \times 10^{-3}\right)$$

x \ y	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.00	0.0000	-0.0010	-0.0020	-0.0030	-0.0040	-0.0050	-0.0060	-0.0070	-0.0081	-0.0091
-0.01	-0.0101	-0.0111	-0.0121	-0.0132	-0.0142	-0.0152	-0.0163	-0.0173	-0.0183	-0.0194
-0.02	-0.0204	-0.0215	-0.0225	-0.0235	-0.0246	-0.0256	-0.0267	-0.0278	-0.0288	-0.0299
-0.03	-0.0309	-0.0320	-0.0331	-0.0341	-0.0352	-0.0363	-0.0374	-0.0385	-0.0395	-0.0406
-0.04	-0.0417	-0.0428	-0.0439	-0.0450	-0.0461	-0.0472	-0.0483	-0.0494	-0.0505	-0.0516
-0.05	-0.0527	-0.0538	-0.0549	-0.0561	-0.0572	-0.0583	-0.0594	-0.0606	-0.0617	-0.0628
-0.06	-0.0640	-0.0651	-0.0662	-0.0674	-0.0685	-0.0697	-0.0708	-0.0720	-0.0732	-0.0743
-0.07	-0.0755	-0.0767	-0.0778	-0.0790	-0.0802	-0.0814	-0.0825	-0.0837	-0.0849	-0.0861
-0.08	-0.0873	-0.0885	-0.0897	-0.0909	-0.0921	-0.0933	-0.0945	-0.0957	-0.0970	-0.0982
-0.09	-0.0994	-0.1006	-0.1019	-0.1031	-0.1043	-0.1056	-0.1068	-0.1081	-0.1093	-0.1106
-0.10	-0.1118	-0.1131	-0.1144	-0.1156	-0.1169	-0.1182	-0.1194	-0.1207	-0.1220	-0.1233
-0.11	-0.1246	-0.1259	-0.1272	-0.1285	-0.1298	-0.1311	-0.1324	-0.1337	-0.1351	-0.1364
-0.12	-0.1377	-0.1391	-0.1404	-0.1417	-0.1431	-0.1444	-0.1458	-0.1471	-0.1485	-0.1499
-0.13	-0.1512	-0.1526	-0.1540	-0.1554	-0.1567	-0.1581	-0.1595	-0.1609	-0.1623	-0.1637
-0.14	-0.1651	-0.1666	-0.1680	-0.1694	-0.1708	-0.1723	-0.1737	-0.1751	-0.1766	-0.1780
-0.15	-0.1795	-0.1810	-0.1824	-0.1839	-0.1854	-0.1868	-0.1883	-0.1898	-0.1913	-0.1928
-0.16	-0.1943	-0.1958	-0.1973	-0.1989	-0.2004	-0.2019	-0.2035	-0.2050	-0.2065	-0.2081
-0.17	-0.2097	-0.2112	-0.2128	-0.2144	-0.2159	-0.2175	-0.2191	-0.2207	-0.2223	-0.2239
-0.18	-0.2255	-0.2272	-0.2288	-0.2304	-0.2321	-0.2337	-0.2354	-0.2370	-0.2387	-0.2404
-0.19	-0.2420	-0.2437	-0.2454	-0.2471	-0.2488	-0.2505	-0.2522	-0.2540	-0.2557	-0.2574
-0.20	-0.2592	-0.2609	-0.2627	-0.2645	-0.2662	-0.2680	-0.2698	-0.2716	-0.2734	-0.2752
-0.21	-0.2770	-0.2789	-0.2807	-0.2825	-0.2844	-0.2863	-0.2881	-0.2900	-0.2919	-0.2938
-0.22	-0.2957	-0.2976	-0.2995	-0.3015	-0.3034	-0.3053	-0.3073	-0.3093	-0.3112	-0.3132
-0.23	-0.3152	-0.3172	-0.3193	-0.3213	-0.3233	-0.3254	-0.3274	-0.3295	-0.3316	-0.3337
-0.24	-0.3358	-0.3379	-0.3400	-0.3421	-0.3443	-0.3464	-0.3486	-0.3508	-0.3530	-0.3552
-0.25	-0.3574	-0.3596	-0.3619	-0.3641	-0.3664	-0.3687	-0.3710	-0.3733	-0.3756	-0.3780
-0.26	-0.3803	-0.3827	-0.3851	-0.3875	-0.3899	-0.3923	-0.3947	-0.3972	-0.3997	-0.4022
-0.27	-0.4047	-0.4072	-0.4098	-0.4123	-0.4149	-0.4175	-0.4201	-0.4227	-0.4254	-0.4281
-0.28	-0.4308	-0.4335	-0.4362	-0.4390	-0.4417	-0.4445	-0.4474	-0.4502	-0.4531	-0.4559
-0.29	-0.4589	-0.4618	-0.4648	-0.4677	-0.4708	-0.4738	-0.4769	-0.4800	-0.4831	-0.4862
-0.30	-0.4894	-0.4926	-0.4959	-0.4991	-0.5024	-0.5058	-0.5091	-0.5125	-0.5160	-0.5195
-0.31	-0.5230	-0.5265	-0.5301	-0.5338	-0.5375	-0.5412	-0.5449	-0.5488	-0.5526	-0.5565
-0.32	-0.5605	-0.5645	-0.5686	-0.5727	-0.5769	-0.5811	-0.5854	-0.5898	-0.5942	-0.5987
-0.33	-0.6033	-0.6079	-0.6126	-0.6174	-0.6223	-0.6273	-0.6324	-0.6376	-0.6428	-0.6482
-0.34	-0.6537	-0.6593	-0.6650	-0.6709	-0.6769	-0.6831	-0.6894	-0.6959	-0.7026	-0.7095
-0.35	-0.7166	-0.7240	-0.7316	-0.7395	-0.7477	-0.7562	-0.7652	-0.7746	-0.7845	-0.7949
-0.36	-0.8061	-0.8181	-0.8311	-0.8454	-0.8614	-0.8798	-0.9022	-0.9324	-	-
-1/e	-1.0000					-				

Table 2: Values of W(z) for the minor branch

$$z = x - \left(y \times 10^{-3}\right)$$

x \ y	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0.00	- ∞	-9.1180	-8.3351	-7.8725	-7.5419	-7.2840	-7.0722	-6.8922	-6.7357	-6.5972
-0.01	-6.4728	-6.3599	-6.2565	-6.1611	-6.0725	-5.9898	-5.9122	-5.8391	-5.7701	-5.7046
-0.02	-5.6423	-5.5830	-5.5262	-5.4719	-5.4198	-5.3696	-5.3214	-5.2749	-5.2300	-5.1865
-0.03	-5.1445	-5.1037	-5.0642	-5.0258	-4.9885	-4.9522	-4.9169	-4.8825	-4.8489	-4.8162
-0.04	-4.7842	-4.7529	-4.7224	-4.6925	-4.6633	-4.6347	-4.6066	-4.5791	-4.5522	-4.5257
-0.05	-4.4998	-4.4743	-4.4492	-4.4247	-4.4005	-4.3767	-4.3533	-4.3304	-4.3077	-4.2854
-0.06	-4.2635	-4.2419	-4.2206	-4.1996	-4.1789	-4.1585	-4.1384	-4.1186	-4.0990	-4.0797
-0.07	-4.0606	-4.0418	-4.0232	-4.0048	-3.9866	-3.9687	-3.9510	-3.9335	-3.9162	-3.8990
-0.08	-3.8821	-3.8654	-3.8488	-3.8324	-3.8162	-3.8001	-3.7843	-3.7685	-3.7530	-3.7375
-0.09	-3.7223	-3.7072	-3.6922	-3.6773	-3.6626	-3.6481	-3.6336	-3.6193	-3.6052	-3.5911
-0.10	-3.5772	-3.5633	-3.5496	-3.5360	-3.5225	-3.5092	-3.4959	-3.4827	-3.4697	-3.4567
-0.11	-3.4439	-3.4311	-3.4184	-3.4059	-3.3934	-3.3810	-3.3687	-3.3565	-3.3443	-3.3323
-0.12	-3.3203	-3.3084	-3.2966	-3.2849	-3.2733	-3.2617	-3.2502	-3.2388	-3.2274	-3.2161
-0.13	-3.2049	-3.1938	-3.1827	-3.1717	-3.1607	-3.1498	-3.1390	-3.1282	-3.1175	-3.1069
-0.14	-3.0963	-3.0858	-3.0753	-3.0649	-3.0546	-3.0443	-3.0340	-3.0239	-3.0137	-3.0036
-0.15	-2.9936	-2.9836	-2.9737	-2.9638	-2.9539	-2.9442	-2.9344	-2.9247	-2.9150	-2.9054
-0.16	-2.8959	-2.8863	-2.8769	-2.8674	-2.8580	-2.8487	-2.8393	-2.8301	-2.8208	-2.8116
-0.17	-2.8025	-2.7933	-2.7842	-2.7752	-2.7662	-2.7572	-2.7482	-2.7393	-2.7304	-2.7216
-0.18	-2.7128	-2.7040	-2.6952	-2.6865	-2.6778	-2.6692	-2.6605	-2.6519	-2.6434	-2.6348
-0.19	-2.6263	-2.6178	-2.6094	-2.6009	-2.5925	-2.5842	-2.5758	-2.5675	-2.5592	-2.5509
-0.20	-2.5426	-2.5344	-2.5262	-2.5180	-2.5099	-2.5017	-2.4936	-2.4855	-2.4774	-2.4694
-0.21	-2.4614	-2.4534	-2.4454	-2.4374	-2.4294	-2.4215	-2.4136	-2.4057	-2.3978	-2.3900
-0.22	-2.3821	-2.3743	-2.3665	-2.3587	-2.3509	-2.3432	-2.3354	-2.3277	-2.3200	-2.3123
-0.23	-2.3046	-2.2969	-2.2892	-2.2816	-2.2740	-2.2663	-2.2587	-2.2511	-2.2435	-2.2360
-0.24	-2.2284	-2.2208	-2.2133	-2.2058	-2.1982	-2.1907	-2.1832	-2.1757	-2.1682	-2.1608
-0.25	-2.1533	-2.1458	-2.1384	-2.1309	-2.1235	-2.1160	-2.1086	-2.1012	-2.0938	-2.0863
-0.26	-2.0789	-2.0715	-2.0641	-2.0567	-2.0493	-2.0419	-2.0345	-2.0271	-2.0197	-2.0123
-0.27	-2.0050	-1.9976	-1.9902	-1.9828	-1.9754	-1.9680	-1.9606	-1.9532	-1.9458	-1.9384
-0.28	-1.9310	-1.9236	-1.9162	-1.9088	-1.9013	-1.8939	-1.8865	-1.8790	-1.8716	-1.8641
-0.29	-1.8566	-1.8492	-1.8417	-1.8342	-1.8267	-1.8191	-1.8116	-1.8041	-1.7965	-1.7889
-0.30	-1.7813	-1.7737	-1.7661	-1.7585	-1.7508	-1.7431	-1.7354	-1.7277	-1.7200	-1.7122
-0.31	-1.7044	-1.6966	-1.6887	-1.6809	-1.6730	-1.6650	-1.6571	-1.6491	-1.6410	-1.6330
-0.32	-1.6248	-1.6167	-1.6085	-1.6003	-1.5920	-1.5837	-1.5753	-1.5669	-1.5584	-1.5499
-0.33	-1.5413	-1.5326	-1.5239	-1.5151	-1.5062	-1.4973	-1.4882	-1.4791	-1.4699	-1.4606
-0.34	-1.4512	-1.4417	-1.4321	-1.4223	-1.4124	-1.4024	-1.3922	-1.3819	-1.3713	-1.3606
-0.35	-1.3497	-1.3386	-1.3272	-1.3155	-1.3035	-1.2912	-1.2786	-1.2654	-1.2518	-1.2376
-0.36	-1.2228	-1.2071	-1.1904	-1.1724	-1.1528	-1.1307	-1.1047	-1.0708	-	-
-1/e	-1.0000					-				

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